Necessary and sufficient condition that a Bratteli-Vershik adic system is uniquely ergodic (Natural extension of arithmetic algorithms and S-adic system)

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By

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Abstract

In this survey we will introduce a necessary and sufficient condition that a Bratteli-Vershik adic system is uniquely ergodic.

§ 1. Introduction

We call a topological dynamical system (X, T) a *Cantor minimal system (Cantor system, or minimal Cantor system)* if X is a Cantor set (i.e., a compact totally disconnected metric space with no isolated points) and T is a homeomorphism on X acting minimally (i.e., every T-orbit is dense in X, or equivalently, the only closed T-invariant sets are X and the empty set.). Cantor minimal systems include many interesting and important dynamical systems: substitution dynamical systems, Toeplitz minimal systems, odometers, Denjoy minimal systems and so on. Each dynamical system has a suitable representation. But Cantor minimal systems have some kind of "universal" representation which is called a *Bratteli-Vershik representation.* For the details of Bratteli-Vershik represention, we recommend the reader to see [HPS].

In this survey we will introduce a necessary and sufficient condition that a Bratteli-Vershik adic system is uniquely ergodic. The original proof is due to R. Gjerde [G]. However we used another (equivalent) condition. (Also see, [Y]).

Basically, we use notations and definitions in [HPS] and [GPS]. Here we list some notations and definitions in this survey. Suppose $\mathcal{B} = (V, E, \geq)$ $(V = \bigsqcup_{n=0}^{\infty} V_n, E = \bigsqcup_{n=1}^{\infty} E_n)$ is a properly ordered (also called simply ordered) Bratteli diagram.

- Let $r : E \to V$ denote the range map and $s : E \to V$ denote the source map. Namely, $e \in E_n$ connects between $s(e) \in V_{n-1}$ and $r(e) \in V_n$.
- Let $X_{\mathcal{B}}$ denote the set of all infinite paths of $\mathcal{B}$. I.e.,

$$X_{\mathcal{B}} := \left\{ (e_n) \in \prod_{n=1}^{\infty} E_n \mid \forall n \in \mathbb{N},\ r(e_n) = s(e_{n+1}) \right\}.$$

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- For $u \in V_k$, let $\mathcal{P}(u)$ denote the set of all finite paths between the unique top vertex $v_0 \in V_0$ and u. Similarly, for $v \in V_n (n > k)$, let $\mathcal{P}(u, v)$ denote the set of all finite paths between u and v.
- Let $(X_{\mathcal{B}}, S_{\mathcal{B}})$ denote the Bratteli-Vershik system of $\mathcal{B}$. Namely, $S_{\mathcal{B}} : X_{\mathcal{B}} \to X_{\mathcal{B}}$ is the Vershik (lexicographic) map defined by the order $\geq$ on E.
- For $u \in V_k$ and $p = (e_1, e_2, \ldots, e_k) \in \mathcal{P}(u)$, let $[p]$ denote *the cylinder set of* p in $X_{\mathcal{B}}$. I.e.,

$$[p] := \{(x_n) \in X_{\mathcal{B}} \mid (x_1, \ldots, x_k) = (e_1, \ldots, e_k)\}$$

It is easy to see that $[p]$ is a clopen (open and closed) set. Let μ denote an $S_{\mathcal{B}}$-invariant Borel probability measure on $X_{\mathcal{B}}$. We define

$$\mu(u) := \mu([p]), \quad \text{where } p \in \mathcal{P}(u).$$

Since μ is $S_{\mathcal{B}}$-invariant, $\mu(u)$ is well-defined (i.e., it is not depend on the choice of $p \in \mathcal{P}(u)$).

§ 2. The proof of theorem

Theorem 2.1. *Let $(X_{\mathcal{B}}, S_{\mathcal{B}})$ be a Bratteli-Vershik adic system. The following are equivalent:*

(1) $(X_{\mathcal{B}}, S_{\mathcal{B}})$ *is uniquely ergodic,*

(2) *for any $k, n \in \mathbb{N}$ $(k < n)$ and $u \in V_k$, $\lim_{n\to\infty} A_n(u) = \lim_{n\to\infty} a_n(u)$ holds, where*

$$A_n(u) := \max_{v \in V_n} \frac{\#\mathcal{P}(u, v)}{\#\mathcal{P}(v)} \quad \textit{and} \quad a_n(u) := \min_{v \in V_n} \frac{\#\mathcal{P}(u, v)}{\#\mathcal{P}(v)}.$$

Proof. **(1)⇒(2):** Let μ be the unique $S_{\mathcal{B}}$-invariant Borel probability measure. Choose any $u \in V_k$ and fix it. It suffices to show

$$\lim_{n\to\infty} A_n(u) = \mu(u) \left(= \lim_{n\to\infty} a_n(u)\right).$$

Note that $\{A_n(u)\}_{n \geq k}$ is a decreasing sequence and bounded from below. So $\lim A_n(u)$ exists. For $A_n(u)$, there exists $x_n = (x_{n,i})_{i \geq 1} \in X_{\mathcal{B}}$ such that

- $A_n(u) = \dfrac{\#\mathcal{P}(u, v_n)}{\#\mathcal{P}(v_n)}$, where $v_n := r(x_{n,n}) \in V_n$,
- the (finite) path $(x_{n,1}, \ldots, x_{n,n})$ is the minimal in $\mathcal{P}(v_n)$.

The unique ergodicity of $(X_{\mathcal{B}}, S_{\mathcal{B}})$ implies that for any $x \in X_{\mathcal{B}}$ and any continuous function f on $X_{\mathcal{B}}$,

$$\lim_{n\to\infty} \frac{1}{n} \sum_{i=0}^{n-1} f(S_{\mathcal{B}}^i x) = \int_{X_{\mathcal{B}}} f \, d\mu \quad \text{uniformly}.$$

(See [W], Theorem 6.19.) Moverover by the compactness of $X_{\mathcal{B}}$, for any sequence $\{y_n\} \subset X_{\mathcal{B}}$ and any increasing sequence $\{a_n\} \subset \mathbb{N}$, we have

$$\lim_{n\to\infty} \frac{1}{a_n} \sum_{i=0}^{a_n-1} f(S_{\mathcal{B}}^i y_n) = \int_{X_{\mathcal{B}}} f \, d\mu. \tag{2.1}$$

Choose any cylinder set $[p]$ ($p \in \mathcal{P}(u)$) and fix it. Let $1_{[p]}$ denote the indicator function defined by $1_{[p]}(x) = 1$ iff $x \in [p]$ and $1_{[p]}(x) = 0$ iff $x \notin [p]$. Then $1_{[p]}$ is continuous because $[p]$ is clopen. We apply (2.1) to $y_n := x_n$, $f := 1_{[p]}$ and $a_n := \#\mathcal{P}(v_n)$. Therefore we have

$$\begin{aligned}
\lim_{n\to\infty} A_n(u) &= \lim_{n\to\infty} \frac{\#\mathcal{P}(u, v_n)}{\#\mathcal{P}(v_n)} \\
&= \lim_{n\to\infty} (\text{relative frequency of paths in } \mathcal{P}(v_n) \text{ which are extensions of } p.) \\
&= \lim_{n\to\infty} \frac{1}{\#\mathcal{P}(v_n)} \sum_{i=0}^{\#\mathcal{P}(v_n)-1} 1_{[p]}(S_{\mathcal{B}}^i x_n) \\
&= \int_{X_{\mathcal{B}}} 1_{[p]} \, d\mu = \mu(u).
\end{aligned}$$

(2)⇒(1): A Borel probability measure μ on a metric space $X_{\mathcal{B}}$ is regular. Namely, for any $\varepsilon > 0$ and any Borel set $B \subset X_{\mathcal{B}}$, there exist an open set O_ε and a closed set C_ε with $C_\varepsilon \subset B \subset O_\varepsilon$ such that $\mu(O_\varepsilon \setminus C_\varepsilon) < \varepsilon$ (See [W] Theorem 6.1). Moreover, the cylinder sets of $X_{\mathcal{B}}$ generates the topology. Therefore we will show that for any $S_{\mathcal{B}}$-invariant Borel probability measures μ, ν and any cylinder set $[p] \subset X_{\mathcal{B}}$,

$$\mu([p]) = \nu([p]).$$

To check the above, it suffices to show that for any $u \in V_k$ and any $S_{\mathcal{B}}$-invariant probability measure μ,

$$\mu(u) = \lim_{n\to\infty} A_n(u) = \lim_{n\to\infty} a_n(u)$$

because this equality means that the measure of the cylinder set $[p]$ with $p \in \mathcal{P}(u)$ is not dependent on μ.

Let $n > k$. The cylinder sets terminating at vertices in V_n are clopen partitions of $X_{\mathcal{B}}$, i.e., $X_{\mathcal{B}} = \bigsqcup_{v\in V_n} \bigsqcup_{p'\in\mathcal{P}(v)} [p']$. We have

$$1 = \sum_{v\in V_n} \#\mathcal{P}(v)\mu(v).$$

For a cylinder set $[p]$ with $p \in \mathcal{P}(u)$, $[p]$ is decomposed into the cylinder sets $[pq]$'s where $q \in \mathcal{P}(u, v)$ and $pq \in \mathcal{P}(v)$. Therefore $[p] = \bigsqcup_{v\in V_n} \bigsqcup_{q\in\mathcal{P}(u,v)} [pq]$ and

$$\mu(u) = \sum_{v\in V_n} \#\mathcal{P}(u, v)\mu(v).$$

Now we use the following inequality: For non-negative sequence $\{a_i\}_{i=1}^n$ and positive sequence $\{b_i\}_{i=1}^n$,

$$\min_{1\le i\le n} \frac{a_i}{b_i} \le \frac{\sum_{i=1}^n a_i}{\sum_{i=1}^n b_i} \le \max_{1\le i\le n} \frac{a_i}{b_i}.$$

Then we have

$$a_n(u) \leq \frac{\mu(u)}{1} = \frac{\sum_{v \in V_n} \#\mathcal{P}(u,v)\mu(v)}{\sum_{v \in V_n} \#\mathcal{P}(v)\mu(v)} \leq A_n(u).$$

This implies that $\lim a_n(u) = \mu(u) = \lim A_n(u)$. $\square$

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References

[G] R. Gjerde, *Bratteli diagrams and dimension groups, Applications to the theory of symbolic dynamical systems,* Ph. D. thesis, Norwegian University of Science and Technology, 1998.

[GPS] T.Giordano, I.F.Putnam and C.F.Skau, *Topological orbit equivalence and C^* crossed products,* J. Reine Angew. Math. **469** (1995), 51–111.

[HPS] R.H.Herman, I.F.Putnam and C.F.Skau, *Ordered Bratteli diagrams,dimension groups, and topological dynamics,* Internat.J.Math. **3** (1992), 827–864.

[W] P.Walters, *An Introduction to Ergodic Theory,* Graduate Texts in Mathematics,vol.79, Springer-Verlag (1981).

[Y] H. Yuasa, *Not finitely but countably Hopf-equivalent clopen sets in a Cantor minimal system,* Topol. Methods Nonlinear Anal. **33** (2009), no. 2, 355–371.